QI and QEC tools for holography

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Abstract

In these notes I review some of the most important aspects of quantum error correction (QEC) which reflect features of holography. Quantum error correction is a subfield of quantum information which has developed in the last twenty years as a necesary tool for realizing quantum computing. To understand quantum error correction, we need to understand its protagonists. Quantum information and quantum noise or errors. Quantum error correction, refers to a set of techniques used to structure quantum information in such a way that it remains accessible and can survive quantum noise (errors). Recently, it has become a relevant tool to describe bulk-to-boundary locality in holography.

QEC is a relatively advanced topic in quantum information and a proper treatment would require dealing with at least three chapters of Nielsen & Chuang's textbook. Nevertheless, I will try to convey some of the most important features in this note.

Classical error correction

In order to explain QEC, it is customary to start with classical error correction. A simple example of a classical code is the repetition code. It has code words

$$0 \rightarrow 000000000 \qquad 1 \rightarrow 111111111. \tag{1}$$

It simply repeats the same piece of information n times. The information is available on any of the n locations. One may see that the loss of any number of up to $n-1$ representatives is recoverable.

$$\square z \square z \square z \square z \square \square \tag{2}$$

The code is called an $[n,1,n]$ code, because it uses n "physical" bits to encode 1 "logical" bit and the minimal number of flips to transform between code-words is n.

Exercise 1. Consider the [7,3,3] Hamming with code words $(b_1, b_2, b_3, b_1 \oplus b_2, b_2 \oplus b_3, b_3 \oplus b_1, b_1 \oplus b_2 \oplus b_3)$. Show that all logical bits (b_1, b_2, b_3) may be recovered if no more than 3 bits are lost.

Hint: Show that the first bit may be recovered from three disjoint sets of physical bits ($d \geq 3$).

Show that a different code word may be obtained by flipping three bits. ($d \leq 3$)

A naïve generalization of this repetition code to quantum mechanics is not possible

$$\psi \rightarrow \psi\psi\psi\psi\psi\psi\psi\psi\psi. \tag{3}$$

If one were provided with a factory producing a desired state ψ, then it would be possible to produce as many copies as desired. However, if one is required to distribute an unknown input state ψ, this same task would not be possible. An obstruction is posed by the so called no cloning theorem, which prohibits producing multiple copies of an unknown quantum state. This is simply a consequence of the linearity of quantum mechanics.

Exercise 2. Prove that there is no linear transformation T such that $T\psi = \psi\psi$ for all $\psi \in \mathbb{C}^d$ (with $d > 1$).

The difficulty of protecting quantum information can thus be atributed to the imposibility of cloning it. This difficulty can be explained in a more useful way by considering complementarity. Whereas, for classical information (bit b) there is only one possible question we may ask "Is b equal to 0 or is it equal to 1?". For a qubit or any piece of piece of **quantum** information, there are in general complementary, non-commuting observables which may not be measured simultaneously. These observables, simply do not allow for compatible measurements, hence the name *complementary*.

Exercise 3. Prove that the standard deviation $\sigma_x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $\sigma_p \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$ for the observables x and p (the cannonical field operators) are such $\sigma_x \sigma_y \geq \hbar/2$. Which states can saturate this relation?

Exercise 4. Consider a spin 1/2 particle (qubit). Do the Pauli spin operators X, Y, Z satisfy the same complementarity relation? If one operator is perfectly defined, what is the variance for the others?

Classical secret sharing

In order to protect an observable associated to a piece of quantum information, complementary observables must be kept unaccessible to the environment. They must be hidden from it, lest the environment measure them. If we want to measure Z and the environment has already measured X then we will only obtaine a random outcome. In nature there is no way to distinguish the environment from another part of the measurment aparatus. For this reason quantum codes need to balance another property. They need to keep quantum information secret!!! In the classical language, these are called secret sharing schemes. They tell us that a certain piece of information only becomes available when a large enough fraction of the physical bits are available.

Example 1. Consider the secret bit b and the random bits $(r_1, r_2, \ldots, r_{n-1})$ if we prepare the bitstring $\left(r_1, r_2, \ldots, r_{n-1}, b\bigoplus_{j=1}^{n-1} r_j\right)$ it will only be possible to learn the value of b if we have access to all the n bits. Note that each string of at most $n-1$ bits is uniformly random and yields no information about the value of b.

This simple secret sharing scheme is a $\langle n, 1, n \rangle$ scheeme since it uses n, "physical shares" (in this case bits) to hide 1 logical bit and all n shares are necesary to recover it. Note that it is awfull as an error correcting code, since the loss of any of the physical bits leads to losing the logical bit.

Example 2. Consider the secret bit b and the random bits (r_1, r_2, r_3) if we prepare the three shares $(\{b \oplus r_1, b \oplus r_1 \oplus r_2\}, \{b \oplus r_2, b \oplus r_2 \oplus r_3\}, \{b \oplus r_3, b \oplus r_3 \oplus r_1\})$, where each share includes two bits of information. This is a $\langle 3, 1, 2 \rangle$ secret sharing scheme since at least 2 shares are necesary to recover the secret bit.

One of the dificulties for secret sharing schemes is keeping the shares small. In the last example, we needed to include two bits per share in order to keep the b secret from any single share. In fact, it is not possible to construct a $\langle 3, 1, 2 \rangle$ secret sharing scheme for a bit where each share is a single bit.

Exercise 5. Construct a $\langle 3, 1, 2 \rangle$ secret sharing scheme for a trit ($t \in \{0, 1, 2\}$) such that each of the three shares ic composedo of a single trit.

Hint: You will only need a single random trit to define the shares.

Quantum error correction

A traditional (subspace) quantum code, uses a subspace (identified by a projector P) of the full physical Hilbert $\mathcal{H}$ space in order to keep quantum information safe from noise. In general, if some noise process does occur, it will the state will change be necesary to identify an appropriate recovery procedure. Since in general, neither noise nor recovery are unitary processes, it becomes convenient to work in the density matrix picture. Operationally, the recovery condition becomes

$$\mathcal{R} \circ \mathcal{N}(P \cdot P) = P \cdot P, \tag{4}$$

where $\mathcal{N}(\rho) := \sum_k N_k \rho N_k^\dagger$ is the noise operator and $\mathcal{R} := \sum_k R_k \rho R_k^\dagger$ is the recovery operator. This representation of the maps $\mathcal{N}$ and $\mathcal{R}$ is called Krauss representation and operators N_k and R_k are called Krauss operators.

Exercise 6. Find a Krauss representation for the local depolarizing noise map $\Delta_S(\rho) := \mathrm{id}_S / d_S \otimes \mathrm{Tr}_S[\rho]$.

If the subsystem S is a known factor of a tensor product Hilbert space, the local depolarizing noise on S is equivalent to the erasure noise in terms of its recoverability.

Knill-Lafflame condition

There is a necessary and sufficient condition for the noise map $\mathcal{N}$ to be correctable. Namely, what is necessary is that $P N_a^\dagger N_b P \propto P$ for all a, b. This is called the Knill-Lafflame condition. We say that a factor of the Hilbert space is correctable, if the depolarizing noise in that factor is correctable.

Given a code subspace P of a factorized Hilbert space $\mathcal{H}$, the **distance** of the code is equal to the minimal number of factors which need to be included in S, such that Δ_S is **not** correctable.

Exercise 7. Construct the three qutrit code and show that the the depolarization (or loss) of any single qubit is correctable (it is a $[[3,1,2]]_3$ QEC where $d = 2$ is the distance). This implies that no information about the code state is available from a single qutrit.

Hint: Use generalized unitary Pauli matrices on qudits given by $Z = \mathrm{diag}(1, \omega, \omega^2)$ and $X = \sum_{j=1}^3 | j+1\rangle\langle j|$ and tensor product thereof to define a three dimensional code space as a subspace which is a $+1$ eigenspace of two such operators. Here where $\omega \equiv e^{i2\pi/3}$.

Write two operators $\bar{X}$ and $\bar{Z}$ with similar anticommutation relations as X and Z which preserve the code subspace. Show that there are operators with the same action in the code subspace which are supported in any pair of the qutrits.

Compare this qutrit code with the Rindler-AdS reconstruction seen in class this morning.

Codes-spaces can also occur as ground spaces of more natural Hamiltonians.

Exercise 8. Consider the ground state subspace of $H = \sum_{\{i,j\} \subset [1,4]} S_i \cdot S_j$ on four qubits. Show that it has dimension 2. Show that it forms a $[[4,1,2]]$ quantum code with respect to the factorization into qubits.

Hint: Conside possible spin states with total angular momentum 0.

Show how it is possible to swap between non-orthgonal basis states by accessing only three of the qubits.

Exercise 9. Consider placing a good quantum code $[[n_1, k_1, d_1]]$ and a bad one $[[n_2, k_2, d_2]]$ side by side. Assuming $d_2 \leq d_1$, we get a code $[[n_1 + n_2, k_1 + k_2, d_2]]$. Does this mean that the first k_1 logical qubits are worse protected now in this code? Discuss.

One of the features of operator algebra quantum error correction (OAQEC) is to focus on a subalgebra of observables. In particular, we may focus on the subalgebra assocaited to the first k_1 qubit and finda that they are better protected than naively expected.

Entropic condition

A purification, is a pure state on a larger Hilbert space such that its reduced density matrix coincides with the desired state. A famous purification in the context of Holography is the thermofield double state.

$$| \Psi_{\mathrm{TFD}}\rangle := \frac{1}{\sqrt{\mathcal{Z}}} \sum_j e^{-\beta \varepsilon_j / 2} | j\rangle | j\rangle \tag{5}$$

This state is manifestly pure and we are assuming that there is a second “ancillary” Hilbert space with the same structure as the original. This state has the property

$$\sigma_\beta := \sum \frac{1}{\mathcal{Z}} \sum_j e^{-\beta \varepsilon_j} | j\rangle\langle j| = \mathrm{Tr}_2[| \Psi_{\mathrm{TFD}}\rangle\langle\Psi_{\mathrm{TFD}}|]. \tag{6}$$

There are many states with this property. In particular, no matter what happens on the second side (any unitary operation or even map is allowed), the reduced density matrix on the first component will continue to be the same.

In holography, it may be difficult to evaluate the Knill-Laflamme condition. The main reason for this is that it is algebraic and we do not have explicit access to what "the code subspace is". Another possibility is to use a sufficient condition for recoverability which is called the Markov condition. In this setting one takes a purification of a mixed state supported on the code space to asses the "average" quality of recovery from the loss of a certain subsystem.

If A is the purification, B the remaining physical subsystem and C is the physical subsystem which is lost, the quantity which needs to be small in order to guarantee recovery is

$$S_{BC} + S_C - S_B \ll 1. \tag{7}$$

If this quantity is much smaller than one, then we may (on average) approximately recover all the information. This is assuming that the ensemble average, for encoded states look like ρ_{BC}.

In general, one is interested in the recovery of a sub-algebra, and not necesarily all of the logical information. In the case one is dealing with a tensor product purification $A = A_1 \otimes A_2$, then the recovery condition for A_1 when C is lost and B remains is

This allows providing approximate recovery guarantees.

$$S_{A_1B} + S_{BC} - S_B - S_{A_1BC} \ll 1. \tag{8}$$

The quantity on the left is non-negative due to **strong subaditivity**. If it is zero, then A_1 can be perfectly recovered from B. If it is much smaller than one, then it can be approximately recovered. Otherwise, we do not know.

Exercise, consider σ_{BC} as a thermals state for a holographic CFT and A is a purification in the sense of thermofield double (fictitious aditional CFT). What does the picture look like depending on the size of C. What does condition (7) tell us if we consider the main contributio of entropy as comming from the Ryu-Takayanagi prescription (Area $\propto$ Entropy). What does this tell us about reconstructing the interior of the black hole?

What does condition (8) tell us about the setting in which we have two black holes which we label as A_1 and A_2? In general, the general application of the Ryu-Takayanagi prescription is a very powerful tool to analyze the way information is structured in holography.

Exercise 10. Discuss the entanglement wedge hypothesis in terms of the error correcting properties it predicts. Explain why it is better described by operator algebra quantum error correction.

Further reading

There are many open questions with respect to what kind of quantum error correction takes place in holography. In particular, it will necesarily be some form of operator algebra quantum error correction. Furthermore we expecta that it will also be approximate and that the notion of code-subspace may need to be reexamined.

1. Almheiri, A., Dong, X., & Harlow, D. (2015). Bulk locality and quantum error correction in AdS/CFT. JHEP, 2015(4), 163. arXiv:1411.7041

2. Pastawski, F., Yoshida, B., Harlow, D., & Preskill, J. (2015). Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence. JHEP, 2015(6), 149. arXiv:1503.06237

3. Freivogel, B., Jefferson, R. A., & Kabir, L. (2016). Precursors, gauge invariance, and quantum error correction in AdS/CFT. JHEP, 2016(4). arXiv:1602.04811

4. Pastawski, F., & Preskill, J. (2017). Code properties from holographic geometries. PRX, 7(2), 21022. arXiv:1612.00017

5. Pastawski, F., Eisert, J., & Wilming, H. (2017). Towards Holography via Quantum Source-Channel Codes. PRL, 119(2), 20501. arXiv:1611.07528

6. Kim, I. H., & Kastoryano, M. J. (2017). Entanglement renormalization, quantum error correction, and bulk causality. JHEP, 2017(4). arXiv:1701.00050

7. Kribs, D. W., Laflamme, R., & Poulin, D. (2005). Unified and Generalized Approach to Quantum Error Correction. PRL, 94(18), 180501-4.

8. Cotler, J., Hayden, P., Salton, G., Swingle, B., & Walter, M. (2017). Entanglement Wedge Reconstruction via Universal Recovery Channels.

9. Bény, C. (2009). Conditions for the approximate correction of algebras.